EXERCISE – V JEE PROBLEMS

1. (a) For all $x \in (0, 1)$ **[JEE 2000 (Scr.), 1 + 1 + 1]**
(A) $e^x < 1 + x$ (B) $\log_e (1 + x) < x$
(C) $\sin x > x$ (D) $\log_e x > x$

(b) Consider the following statement S and R
S : Both sin x and cos x are decreasing functions in the interval $\left(\frac{\pi}{2}, \pi\right)$
R : If a differentiable function decreases in an interval (a, b) then its derivative also decreases in (a, b).
Which of the following is true ?
(A) both S and R are wrong
(B) both S and R are correct, but R is not the correct explanation for S
(C) S is correct and R is the correct explanation for S
(D) S is correct and R is wrong

(c) Let $f(x) = \int e^x (x - 1)(x - 2)\,dx$ then f decreases in the interval
(A) $(-\infty, 2)$ (B) $(-2, -1)$ (C) $(1, 2)$ (D) $(2, +\infty)$

2. (a) If $f(x) = x\,e^{x(1-x)}$, then f(x) is **[JEE 2001, 1 + 5]**
(A) increasing on $(-1/2, 1)$ (B) decreasing on $[-1/2, 1]$
(C) increasing on R (D) decreasing on R

(b) Let $-1 \le p \le 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it.

3. The length of the longest interval in which the function $f(x) = 3 \sin x - 4 \sin^3 x$ is increasing, is
[JEE 2002 (Scr.), 3]
(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{2}$ (D) π

4. (a) Using the relation $2(1 - \cos x) < x^2$, $x \ne 0$ or otherwise, prove that $\sin(\tan x) \ge x$, $\forall x \in [0, \pi/4]$.
[JEE 2003 (Mains), 4 + 4]

(b) Let $f : [0, 4] \to R$ be a differentiable function.
(i) Show that there exist
$a, b \in [0, 4]$, $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$
(ii) Show that there exist α, β with $\alpha, \beta \in (0, 2)$ such that
$$\int_0^4 f(t)\,dt = 2(\alpha f(\alpha^2) + \beta f(\beta^2))$$

5. (a) Let $f(x) = \begin{cases} x^\alpha \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if α equals
[JEE 2004 (Scr.)]
(A) –2 (B) –1 (C) 0 (D) 1/2

(b) If f is a strictly increasing function, then $\lim_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is equal to
(A) 0 (B) 1 (C) –1 (D) 2

6. If $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$, using Rolle's theorem, prove that at least one root of p(x) lies between $(45^{1/100}, 46)$. **[JEE 2004, 2]**

7. If f(x) is a twice differentiable function and given that f(1) = 1, f(2) = 4, f(3) = 9, then
[JEE 2005 (Scr.), 3]
(A) $f''(x) = 2$, for $\forall\ x \in (1, 3)$
(B) $f''(x) = f'(x) = 2$, for some $x \in (2, 3)$
(C) $f''(x) = 3$, for $\forall\ x \in (2, 3)$
(D) $f''(x) = 2$, for some $x \in (1, 3)$

8. (a) Let f(x) = 2 + cos x for all real x. **[JEE 2007, 3]**
Statement-1 : For each real t, there exists a point 'c' in $[t, t + \pi]$ such that $f'(c) = 0$.
because
Statement-2 : $f(t) = f(t + 2\pi)$ for each real t.
(A) Statement-1 is true, statement-2 is true; statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

(b) Paragraph **[JEE 2007, 4 + 4 + 4]**
If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation f(x) = 0 has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation f(x) = 0 has a root in R.
Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

(i) The line y = x meets $y = ke^x$ for $k \le 0$ at
(A) no point (B) one point
(C) two points (D) more than two points

(ii) The positive value of k for which $ke^x - x = 0$ has only one root is
(A) 1/e (B) 1 (C) e (D) $\log_e 2$

(iii) For k > 0, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is
(A) (0, 1/e) (B) (1/e, 1) (C) (1/e, ∞) (D) (0, 1)

(c) Match the column.
In the following [x] denotes the greatest integer less than or equal to x. **[JEE 2007, 6]**

Column–I	Column–II
(A) $x\,\lvert x\rvert$	(P) continuous in (–1, 1)
(B) $\sqrt{\lvert x\rvert}$	(Q) differentiable in (–1, 1)
(C) x + [x]	(R) strictly increasing in (–1, 1)
(D) $\lvert x-1\rvert + \lvert x+1\rvert$	(S) non differentiable at least at one point in (–1, 1)

9. (a) Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is **[JEE 2008, 3 + 4]**
(A) even and is strictly increasing in (0, ∞)
(B) odd and is strictly decreasing in (–∞, ∞)
(C) odd and is strictly increasing in (– ∞, ∞)
(D) neither even nor odd, but is strictly increasing in (–∞, ∞)

(b) Let f(x) be a non-constant twice differentiable function defined on (–∞, ∞) such that f(x) = f(1 – x) and $f'(1/4) = 0$. Then
(A) $f''(x)$ vanishes at least twice on [0, 1]
(B) $f'(1/2) = 0$ (C) $\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\,dx = 0$
(D) $\int_0^{1/2} f(t)e^{\sin \pi t}\,dt = \int_{1/2}^{1} f(1-t)e^{\sin \pi t}\,dt$

10. For the function $f(x) = x\cos\frac{1}{x}$, $x \ge 1$, **[JEE 2009, 4]**
(A) for at least one x in the interval [1, ∞), f(x + 2) – f(x) < 2
(B) $\lim_{x\to\infty} f'(x) = 1$
(C) for all x in the interval [1, ∞), f(x + 2) – f(x) > 2
(D) $f'(x)$ is strictly decreasing in the interval [1, ∞)

11. Let f be a real valued function defined on the interval (0, ∞) by $f(x) = \ell n\, x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is/are true ? **[JEE 2010, 3]**
(A) $f''(x)$ exists for all $x \in (0, \infty)$
(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on (0, ∞) but not differentiable on (0, ∞).
(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$
(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$

Paragraph for Question Nos. 12 to 13

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in$ IR, and let

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell n\, t\right) f(t)\,dt \text{ for all } x \in (1, \infty)$$

12. Which of the folloiwng is true ? **[JEE 2012]**
(A) g is increasing on (1, ∞)
(B) g is decreasing on (1, ∞)
(C) g is increasing on (1, 2) and decreasing on (2, ∞)
(D) g is decreasing on (1, 2) and increasing on (2, ∞)

13. Consider the statements :
P : There exists some x ∈ IR such that
$f(x) + 2x = 2(1 + x^2)$
Q: There exists some x ∈ IR such that
$2f(x) + 1 = 2x(1 + x)$
Then
(A) both **P** and **Q** are true (B) **P** is true and **Q** is false
(C) **P** is false and **Q** is true (D) both **P** and **Q** are false